Cotangent cohomology of Stanley-Reisner rings

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Abstract

Simplicial complexes X provide commutative rings $A(X)$ via the Stanley-Reisner construction. We calculated the cotangent cohomology, i.e., T^1 and T^2 of $A(X)$ in terms of X. These modules provide information about the deformation theory of the algebro geometric objects assigned to X.

1 Introduction

Denote $[n] := \{0, \ldots, n\}$ and let $\Delta_n := 2^{[n]}$ be the full simplex. A simplicial complex $X \subseteq \Delta_n$ with vertex set $[X] \subseteq [n]$ gives rise to an ideal

$$I_X := \langle \mathbf{x}^p \mid p \in \Delta_n \setminus X \rangle \subseteq \mathbb{C}[x_0, \ldots, x_n] =: P.$$

The *Stanley-Reisner ring* is then $A_X = P/I_X$. We can associate the schemes $\mathbb{A}(X) = \operatorname{Spec} A_X$ and $\mathbb{P}(X) = \operatorname{Proj} A_X$ with these rings. The latter looks like X itself – its simplices have just been replaced by projective spaces.

For each $\mathbb{C}$-algebra A, there is a cohomology theory providing modules T^i_A, see e.g. [And74] or [Lau79]. However, only T^1_A and T^2_A are relevant for the deformation theory of $\operatorname{Spec} A$. The module T^1_A collects the infinitesimal deformations of A and T^2_A contains the obstructions for lifting these deformations to decent parameter spaces. Eventually, $\operatorname{Der}_{\mathbb{C}}(A, A)$ will be called T^0_A.

The main result of the present paper is Theorem 9; it provides the modules T^i ($i = 1, 2$) for Stanley-Reisner rings A_X in terms of the geometry of the original simplicial complex X: The T^i are $\mathbb{Z}^{[n]}$-graded, i.e., each degree $\mathbf{c}$ corresponds to a monomial of the quotient field of the ambient ring P. Splitting $\mathbf{c} = \mathbf{a} - \mathbf{b}$ in its positive and negative part, one obtains disjoint subsets $a, b \subseteq [n]$ as their respective supports. They give rise to certain subsets $N_{a-b}, \widetilde{N}_{a-b} \subseteq X$, cf. Section 2 (right before Lemma 3), and it is the cohomology of their geometric realizations which provides the homogeneous part $T^i_{\mathbf{c}}$ of T^i.

One has to be a little careful with the geometric realization of a subset $N \subseteq X$ which is not necessarily a subcomplex; in particular it depends on whether $\emptyset \in N$ or not – see Section 3 for the definition. In Theorem 13, we present a version of our

T^i-formula that uses only open subsets of X or certain nice subcomplexes. We have chosen a non-trivial example (introduced as Example 1 in Section 2) to illustrate the theory. In particular, it is spread (in eight parts) throughout the text.

In the case that $|X|$ is a homological sphere, Ishida and Oda have proven Theorem 9 in [IO81] using torus embeddings. Moreover, in [Sym97], Symms computes $\mathrm{Hom}(I_X, A_X)$ and T_0^2 when $|X|$ is a 2-dimensional manifold possibly with boundary. Our method is straightforward and allows us to get the T^i for all Stanley-Reisner algebras. We also compute the cup product $T^1 \times T^1 \to T^2$ and the localization maps. Theorem 15 states that they are always injective.

Information about the T^i, the cup product, and their behavior under localization makes it possible to investigate the deformation theory of $\mathbb{A}(X)$ and $\mathbb{P}(X)$. In fact, this paper was originally motivated by a question from Sorin Popescu about the smoothability of $\mathbb{P}(X)$ when $|X| \approx S^n$ as this would have applications for degenerations of Calabi-Yau manifolds. In the forthcoming paper [AC02] we apply our results to the case when X is a combinatorial manifold, e.g., a sphere. Here we can give very explicit results and a good understanding of the deformations of $\mathbb{A}(X)$ and $\mathbb{P}(X)$.

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2 Cotangent cohomology in terms of the complex

Notation. We will often work in the polynomial ring $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_0, \ldots, x_n]$. Monomials are written as $\mathbf{x}^{\mathbf{p}} \in \mathbb{C}[\mathbf{x}]$ with exponents $\mathbf{p} \in \mathbb{N}^{n+1}$. The support of $\mathbf{p}$ is defined as $p := \{i \in [n] \mid \mathbf{p}_i \neq 0\}$. On the other hand, subsets $p \subseteq [n]$ will allways be identified with their characteristic vector $\mathbf{p} \in \{0,1\}^{n+1}$.

Let $X \subseteq \Delta_n$ be a simplicial complex and denote by $\{e_p \mid p \in \Delta_n \setminus X\}$ a basis for $P^{|\Delta_n \setminus X|}$ parametrizing the generators $\mathbf{x}^p$ of I_X. The generating relations among them are

$$R_{p,q} := \quad \mathbf{x}^{q \setminus p}\, e_p - \mathbf{x}^{p \setminus q}\, e_q \,.$$

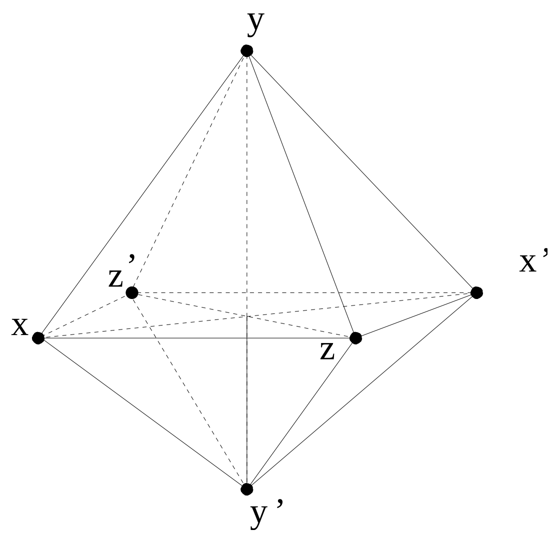
The relations among these relations are

$$R_{p,q,r} : \quad \mathbf{x}^{r \setminus (p \cup q)}\, e_{p,q} - \mathbf{x}^{q \setminus (p \cup r)}\, e_{p,r} + \mathbf{x}^{p \setminus (q \cup r)}\, e_{q,r} \,.$$

Remark. What we have just described is a special case of the so called Taylor resolution – a construction of a free, but in general not minimal, resolution of any monomial ideal. For a description and proof of exactness see e.g. [BPS98].

Example 1. The following simplicial complex D will serve as a running example throughout the text: With vertex set $\{x, y, z, x', y', z'\}$, we define $D \subseteq \Delta_5$ to be the

union of the octahedron with the 8 maximal faces $(x^{(\prime)}y^{(\prime)}z^{(\prime)})$ and the 3 diagonals (xx'), (yy'), and (zz'). Hence, the set $\Delta_5 \setminus D$ providing the generators of I_D consists of all $p \subseteq \{x, y, z, x', y', z'\}$ with $\#p \geq 3$ and containing at least one letter twice.



In general, for a finitely generated $\mathbb{C}$-algebra A, the modules T^i_A allow the following ad hoc definitions: Let $P = \mathbb{C}[\mathbf{x}]$ mapping onto A so that $A \simeq P/I$ for an ideal I. Then T^1_A is the cokernel of the natural map $\mathrm{Der}_{\mathbb{C}}(P, P) \to \mathrm{Hom}_P(I, A)$. Moreover, if

$$0 \to R \to P^m \xrightarrow{j} P \to A \to 0$$

is an exact sequence presenting A as a P module and $R_0 := \langle j(f)e - j(e)f \mid e, f \in P^m \rangle \subseteq R$ denotes the so-called Koszul relations, then R/R_0 is an A module and we obtain T^2_A as the cokernel of the induced map $\mathrm{Hom}_P(P^m, A) \to \mathrm{Hom}_A(R/R_0, A)$.

If $A = A_X$ is a Stanley-Reisner ring, then A_X itself, its resolution, and all interesting A_X-modules such as the T^i_A are $\mathbb{Z}^{n+1}$-graded; just set $\deg e_p = p$, $\deg R_{p,q} = p \cup q$, and $\deg R_{p,q,r} = p \cup q \cup r$. For an element $\mathbf{c} \in \mathbb{Z}^{n+1}$, we denote by

$$\mathrm{Hom}(I_X, A_X)_{\mathbf{c}} \quad \text{and} \quad T^i_{\mathbf{c}}(X) := T^i_{A_X,\mathbf{c}}$$

the homogeneous summands of the corresponding modules. Let $\mathbf{c} = \mathbf{a} - \mathbf{b}$ be the decomposition of $\mathbf{c}$ in its positive and negative part, i.e., $\mathbf{a}, \mathbf{b} \in \mathbb{N}^{n+1}$ with both elements having disjoint supports a and b, respectively. This gives rise to the sets

$$M_{a-b} := \{p \in (\Delta_n \setminus X) \mid (p \cup a) \setminus b \in X\}$$

and

$$M^{(2)}_{a-b} := \{(p, q) \in M_{a-b} \times M_{a-b} \mid (p \cup q \cup a) \setminus b \in X\}.$$

Example 1.2 (continued) Let $p := (xx'yy')$ and $q := (xyy'z)$. Then $p, q \in M_{\emptyset-(yy')}$, but $(p, q) \notin M^{(2)}_{\emptyset-(yy')}$. Moreover, $(xx'yz), (xyz) \notin M_{\emptyset-(yy')}$, but for different reasons.

Lemma 2. *Let* $\mathbf{c} = \mathbf{a} - \mathbf{b}$ *as before. The modules* $\mathrm{Hom}(I_X, A_X)_{\mathbf{c}}$ *and* $T^2_{\mathbf{c}}(X)$ *vanish unless* $\mathbf{b} \in \{0,1\}^{n+1}$*, i.e.,* $\mathbf{b} = b$*. Assuming* $\mathbf{b} = b$*, these modules only depend on the supports* a, b*.*

(i) $\mathrm{Hom}(I_X, A_X)_{\mathbf{c}} = \{\mu : M_{a-b} \to \mathbb{C} \mid \mu(p) = 0 \text{ if } b \not\subseteq p,$
$\mu(p) = \mu(q) \text{ if } (p,q) \in M^{(2)}_{a-b}\}$.

(ii) *Elements of* $\mathrm{Hom}(I_X, A_X)_{\mathbf{c}}$ *yield trivial deformations, i.e., belong to the image of* $\mathrm{Der}_{\mathbb{C}}(P,P)_{\mathbf{c}}$*, iff* $\#(b) = 1$ *and* $\mu(p)$ *is a constant function.*

(iii) $T^2_{\mathbf{c}}(X)$ *is the factor of*

$$\{\mu : M^{(2)}_{a-b} \to \mathbb{C} \mid \mu \text{ is antisymmetric, } \mu(p,q) = 0 \text{ if } (p \cap q) \cup ((p \cup q \cup a) \setminus b) \in X \\ \text{or if } b \not\subseteq p \cup q,\; \mu(p,q) - \mu(p,r) + \mu(q,r) = 0 \text{ if } (p \cup q \cup r \cup a) \setminus b \in X\}$$

by the subspace of functions $\mu(p,q) = \mu'(p) - \mu'(q)$ *with* $\mu'(m) = 0$ *if* $b \not\subseteq m$*.*

Proof. An element $\varphi \in \mathrm{Hom}(I_X, A_X)_{\mathbf{c}}$ maps the generating monomials $\mathbf{x}^p$ to some $\mu(p)\,\mathbf{x}^{p+\mathbf{a}-\mathbf{b}}$ with $\mu(p) \in \mathbb{C}$. The condition that $(p + \mathbf{a} - \mathbf{b}) \in \mathbb{N}^{n+1}$ yields that $\mu(p) = 0$ unless $\mathbf{b} = b$ and $b \subseteq p$. On the other hand, if $\mathbf{x}^{p+\mathbf{a}-\mathbf{b}} \in I_X$, then $\varphi(\mathbf{x}^p) = 0$, and the value $\mu(p)$ does not matter at all. Hence, we may restrict the knowledge of μ to $M_{a-b} \subseteq (\Delta_n \setminus X)$. The linearity of φ translates into the last condition in (i). Eventually, the trivial deformations are spanned by $\varphi = \partial/\partial x_i$.
(iii) One obtains the description of $\mathrm{Hom}_A(R/R_0, A_X)_{\mathbf{c}}$ with the same arguments. We should only remark that it is the Koszul relations $\mathbf{x}^q e_p - \mathbf{x}^p e_q \in R_0$ that are responsible for the vanishing of $\mu(p,q)$ in case of $(p \cap q) \cup ((p \cup q \cup a) \setminus b) \in X$. Afterwards, to get T^2, one needs to divide out the canonical generators $D_m \in \mathrm{Hom}_{\mathbb{C}[\mathbf{x}]}(\mathbb{C}[\mathbf{x}]^{\Delta_n \setminus X}, A_X)$ $(m \in \Delta_n \setminus X)$. They have degree $-m$, and applied to $R_{p,q}$, they yield non-trivial values $\mathbf{x}^{q \setminus p}$ and $-\mathbf{x}^{p \setminus q}$ only if $p = m$ or $q = m$, respectively. Hence, in degree $\mathbf{c}$, the map $\mathbf{x}^{m+\mathbf{a}-b} D_m$ yields $\mu(p,q) = \mu_m(p) - \mu_m(q)$ with μ_m denoting the characteristic function of m. On the other hand, this contribution requires $b \subseteq m$. $\square$

Of course, we are building some sort of cohomology to describe the graded pieces of $T^i(X)$. However, in the previous lemma, the Koszul condition does not seem to fit. Surprisingly, this problem will be overcome by performing kind of a $\mathbf{c}$-shift. Let

$$N_{a-b}(X) := \{f \in X \mid a \subseteq f,\; f \cap b = \emptyset,\; f \cup b \notin X\}$$

and

$$\widetilde{N}_{a-b}(X) := \{f \in N_{a-b} \mid \exists\, b' \subset b \text{ with } f \cup b' \notin X\}.$$

Lemma 3. *Let* $\Phi : M_{a-b} \to X$ *be the application* $\Phi(p) = (p \cup a) \setminus b$*.*

(i) $\mathrm{Im}\,\Phi = N_{a-b}$*. Moreover, an element* $f \in N_{a-b}$ *has a pre-image that does not contain* b *if and only if* $f \in \widetilde{N}_{a-b}$*.*

(ii) *If* $f, g \in N_{a-b}$ *and* $f \cup g \in X$*, then* $f \cup g \in N_{a-b}$*. If, moreover,* $g \in \widetilde{N}_{a-b}$*, then we even obtain* $f \cup g \in \widetilde{N}_{a-b}$*.*

Proof. (i) If $p \in M_{a-b}$, then $\Phi(p) \cup b \supseteq p \notin X$, hence $\Phi(M_{a-b}) \subseteq N_{a-b}$. On the other hand, let $f \in N_{a-b}$ with $f \cup b' \notin X$ for some $b' \subseteq b$. It follows that $f \cup b' \in M_{a-b}$ and $\Phi(f \cup b') = f$. With $b' := b$, this implies $N_{a-b} \subseteq \Phi(M_{a-b})$; with b' being a proper subset of b, we obtain the $\widetilde{N}_{a-b}$ statement. The claims in (ii) are obvious. $\square$

Example 1.3 (continued) Considering the previous $p = (xx'yy')$ and $q = (xyy'z)$ from $M_{\emptyset-(yy')}$, we obtain $\Phi(p) = (xx') \in \widetilde{N}_{\emptyset-(yy')}(D)$ (note that $\Phi(p) = \Phi(xx'y)$), but $\Phi(q) = (xz) \in N_{\emptyset-(yy')}(D) \setminus \widetilde{N}_{\emptyset-(yy')}(D)$.

The map $\Phi : M_{a-b} \twoheadrightarrow N_{a-b}$ of the previous lemma can easily be extended to pairs. That is, with $N^{(2)}_{a-b} := \{(f,g) \in N_{a-b} \times N_{a-b} \mid f \cup g \in X\}$, we also have a surjective application $\Phi : M^{(2)}_{a-b} \twoheadrightarrow N^{(2)}_{a-b}$.

Proposition 4. *Let* $\mathbf{c} = \mathbf{a} - \mathbf{b}$ *as before. The modules* $\operatorname{Hom}(I_X, A_X)_{\mathbf{c}}$ *and* $T^2_{\mathbf{c}}(X)$ *vanish unless* $\mathbf{b} \in \{0,1\}^{n+1}$*, i.e.,* $\mathbf{b} = b$*. Assuming* $\mathbf{b} = b$*, these modules only depend on the supports* a, b*.*

(i) $\operatorname{Hom}(I_X, A_X)_{\mathbf{c}} = \{\lambda : N_{a-b} \to \mathbb{C} \mid \lambda(f) = 0 \text{ if } f \in \widetilde{N}_{a-b},$
$\lambda(f) = \lambda(g) \text{ if } f \cup g \in X\}$.

(ii) *Elements of* $\operatorname{Hom}(I_X, A_X)_{\mathbf{c}}$ *yield trivial deformations, i.e., belong to the image of* $\operatorname{Der}_{\mathbb{C}}(P,P)_{\mathbf{c}}$*, iff* $\#(b) = 1$ *and* $\lambda(f)$ *is a constant function.*

(iii) $T^2_{\mathbf{c}}(X)$ *is the factor of the vector space of the antisymmetric maps* $\lambda : N^{(2)}_{a-b} \to \mathbb{C}$ *such that*

$$\lambda(f,g) = 0 \text{ if } f, g \in \widetilde{N}_{a-b} \text{ and } \lambda(f,g) - \lambda(f,h) + \lambda(g,h) = 0 \text{ if } (f \cup g \cup h) \in X$$

by the subspace $\{\lambda(f) - \lambda(g)\}$ *with* $\lambda = 0$ *on* $\widetilde{N}_{a-b}$*.*

Proof. Denote, just for this proof, the spaces given by (i) and (iii) of the previous proposition by $\operatorname{Hom}(N)$ and $T^2(N)$, respectively. Then we have to ascertain that pulling back via Φ induces isomorphisms $\Phi^* : \operatorname{Hom}(N) \xrightarrow{\sim} \operatorname{Hom}(M)$ and $\Phi^* : T^2(N) \xrightarrow{\sim} T^2(M)$ with $\operatorname{Hom}(M)$ and $T^2(M)$ being the corresponding spaces from Lemma 2.

Step 1. The maps Φ^* *are correctly defined:*
This is clear for the Hom case. For T^2, we set $\mu(p,q) := \lambda(\Phi p, \Phi q)$, and the only non-trivial task is to check the two conditions that should lead to the vanishing of $\mu(p,q)$. If $b \not\subseteq p \cup q$, then both $b \not\subseteq p$ and $b \not\subseteq q$, hence $\Phi(p), \Phi(q) \in \widetilde{N}$ by Lemma 3(i), hence $\lambda(\Phi p, \Phi q) = 0$. On the other hand, if $(p \cap q) \cup ((p \cup q \cup a) \setminus b) \in X$, then we also obtain $b \not\subseteq p$ and $b \not\subseteq q$: Otherwise, if $b \subseteq p$, it would follow that $q \cap b \subseteq p \cap q$ and $q \setminus b \subseteq (p \cup q \cup a) \setminus b$, thus, $q = (q \cap b) \cup (q \setminus b) \subseteq (p \cap q) \cup ((p \cup q \cup a) \setminus b) \in X$, but this contradicts $q \in M$.

Step 2. The maps Φ^* *are injective:*
The Hom case follows from the surjectivity of Φ. For T^2, assume that $\lambda(\Phi p, \Phi q) = \mu(p,q) = \mu(p) - \mu(q)$. In particular, if p, q belong to a common fiber $\Phi^{-1}(f)$, then $\mu(p) - \mu(q) = 0$. Hence, $\mu(p), \mu(q)$ only depend on $\Phi(p)$ and $\Phi(q)$.

Step 3. The maps Φ^* are surjective:
Let $\{\mu(p)\}$ represent an element of $\operatorname{Hom}(M)$. Then, the property that $\mu(p) = \mu(q)$ for $(p, q) \in M^{(2)}$ implies that $\mu(p)$ only depends on $\Phi(p)$. In particular, $\{\mu(p)\} \in \Phi^*(\operatorname{Hom}(N))$.

To check the T^2 case, we would like to proceed similarily with elements $\{\mu(p, q)\} \in T^2(M)$. However, this requires a correction by coboundaries: By the cocycle property of the $\mu(p, q)$'s, we have to find $\{\mu(p)\}$ such that $\tilde{\mu}(p, q) := \mu(p, q)+(\mu(p)-\mu(q))$ vanishes if p, q belong to a common fiber $\Phi^{-1}(f)$. Using the cocycle property again, we see that $\mu(p) := \mu(m_f, p)$ will almost do the job for any fixed $m_f \in \Phi^{-1}(f)$; but we also have to ensure that $\mu(p) = 0$ whenever $b \not\subseteq p$. This is done by proving the following
Claim: Let $m, p \in M$ with $b \not\subseteq m$, $b \not\subseteq p$, and $\Phi(m) \supseteq \Phi(p)$. Then $\mu(m, p) = 0$.
With $f := \Phi(m)$, we have that $(m \cap p) \cup f = (m \cap p) \cup ((m \cup p \cup a) \setminus b)$. Thus, $(m \cap p) \cup f \notin X$ (or the Koszul condition in Lemma 2(iii) immediatly implies $\mu(m, p) = 0$). In particular, $(m \cap p) \cup f \in \Phi^{-1}(f)$, and since $b \not\subseteq m \cup [(m \cap p) \cup f]$ and $b \not\subseteq [(m \cap p) \cup f] \cup p$, we obtain that

$$\mu(m, p) \;=\; \mu(m,\, (m \cap p) \cup f) + \mu((m \cap p) \cup f,\, p) \;=\; 0 + 0 \;=\; 0\,.$$

Eventually, it is possible to define $\lambda(\Phi p, \Phi q) := \tilde{\mu}(p, q) = \mu(m_{\Phi(p)}, m_{\Phi(q)})$, and it remains to show its vanishing for $\Phi(p), \Phi(q) \in \widetilde{N}$. Since, by Lemma 3(ii), $\Phi(p) \cup \Phi(q) \in \widetilde{N}$, we may assume that $\Phi(p) \subseteq \Phi(q)$. Now everything follows from applying the previous claim again. $\square$

Definition 5. A subset $Y \subseteq X$ of a simplicial complex X has property U, or is a U subset, if

$$f, g \in Y \text{ and } f \cup g \in X \Rightarrow f \cup g \in Y\,.$$

If Y has this property, then we define the sets

$$Y^{(k)} := \{(f_0, \ldots, f_k) \in Y^{k+1} \mid f_0 \cup \cdots \cup f_k \in Y\}$$

and the complex of $\mathbb{C}$-vector spaces

$$K^k(Y) := \{\lambda : Y^{(k)} \to \mathbb{C} \mid \lambda \text{ is alternating}\} \subseteq \Lambda^{k+1}\big(\mathbb{C}^Y\big)$$

with the usual differential $d : K^{k-1}(Y) \to K^k(Y)$ defined by

$$d(\lambda)(f_0, \ldots, f_k) := \sum_{v=0}^{k} (-1)^v \lambda(f_0, \ldots, \hat{f}_v, \ldots, f_k)\,.$$

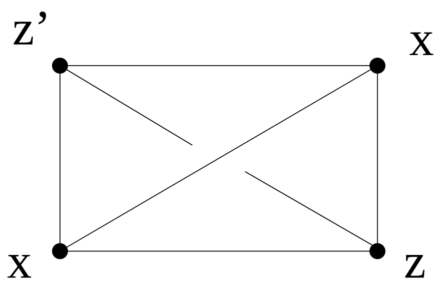
By Lemma 3(ii), both N_{a-b} and $\widetilde{N}_{a-b}$ are U subsets. Moreover, there is a canonical surjection of complexes $K^\bullet(N_{a-b}) \twoheadrightarrow K^\bullet(\widetilde{N}_{a-b})$ leading to

Corollary 6. *Assume* $\mathbf{c} = \mathbf{a} - b$ *with disjoint* $a, b \in X$*. Then*

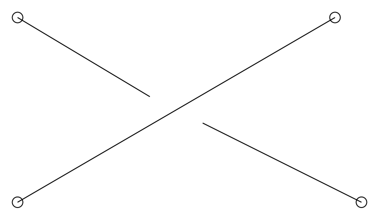
$$\begin{aligned}\operatorname{Hom}(I_X, A_X)_{\mathbf{c}} &\simeq H^0(\ker(K^\bullet(N_{a-b}) \twoheadrightarrow K^\bullet(\widetilde{N}_{a-b})))\\ T^2_{\mathbf{c}}(X) &\simeq H^1(\ker(K^\bullet(N_{a-b}) \twoheadrightarrow K^\bullet(\widetilde{N}_{a-b})))\,,\end{aligned}$$

and the trivial deformations inside $\operatorname{Hom}(I_X, A_X)_{\mathbf{c}}$*, i.e., those yielding* 0 *in* $T^1_{\mathbf{c}}(X)$*, form a one-dimensional subspace whenever* $\#(b) = 1$ *(and are absent otherwise).*

Example 1.4 (continued) We still consider the degree $a = \emptyset$, $b = \{y, y'\}$ for D being the octahedron with diagonals. The set $N_{\emptyset-(yy')}(D)$ consists of the 4 vertices $x^{(')}$, $z^{(')}$ and all the 6 edges connecting them. However, only the interior of the edges xx' and zz' survive in $\widetilde{N}_{\emptyset-(yy')}(D)$.



$N_{\emptyset-(yy')}$



$\tilde{N}_{\emptyset-(yy')}$

To obtain elements of $\operatorname{Hom}(I_D, A_D)_{\mathbf{c}}$, we have to consider maps $\lambda : N \to \mathbb{C}$, i.e., each of the 4 vertices and 6 edges will be assigned a value. The two conditions encoded by "H^0" and "ker" in the previous corollary mean that λ has to be both constant along the graph and zero on $\operatorname{int} xx'$ and $\operatorname{int} zz'$. Hence, $\operatorname{Hom}(I_D, A_D)_{\mathbf{c}} = 0$.

Remark. As we mentioned in the beginning, there is a general cohomological definition of the T^i. Hence, it is no surprise that we ended up with a cohomological description of these spaces in terms of X, too. Moreover, it should even be a challange to find a *direct* way to obtain the previous result (without touching elements). If this involved a description of the so-called cotangent complex, one would obtain important information about the deformation theory of both $\mathbb{A}(X)$ and $\mathbb{P}(X)$.

3 Cotangent cohomology and the geometry of X

In the following, we will relate the previous description of $T^i(X)$ with the geometry of the complex. Let us start with some notation. For $g \subseteq [n]$, denote by $\bar{g} := 2^g$ and $\partial g := \bar{g} \setminus \{g\}$ the full simplex and its boundary, respectively. The *join* $X * Y$ of two complexes X and Y is the complex defined by

$$X * Y := \{f \vee g : f \in X,\ g \in Y\}$$

where $\vee$ means the disjoint union. If $f \in X$ is a face, we may define

- the *link* of f in X; $\operatorname{lk}(f,X) := \{g \in X : g \cap f = \emptyset \text{ and } g \cup f \in X\}$,
- the *open star* of f in X; $\operatorname{st}(f,X) := \{g \in X : f \subseteq g\}$, and
- the *closed star* of f in X; $\overline{\operatorname{st}}(f,X) := \{g \in X : g \cup f \in X\}$.

Notice that the closed star is the subcomplex $\overline{\operatorname{st}}(f,X) = \bar{f} * \operatorname{lk}(f,X)$. Recall that the *geometric realization* of X, denoted $|X|$, may be described by

$$|X| = \left\{\alpha : [n] \to [0,1] \,\middle|\, \{i \mid \alpha(i) \neq 0\} \in X \text{ and } \textstyle\sum_i \alpha(i) = 1\right\}.$$

To every non-empty $f \in X$, one assigns the *relatively open* simplex $\langle f \rangle \subseteq |X|$;

$$\langle f \rangle = \{\alpha \in |X| \mid \alpha(i) \neq 0 \text{ if and only if } i \in f\}.$$

On the other hand, each subset $Y \subseteq X$ determines a topological space

$$\langle Y \rangle := \begin{cases} \bigcup_{f \in Y} \langle f \rangle & \text{if } \emptyset \notin Y, \\ \operatorname{cone}\left(\bigcup_{f \in Y} \langle f \rangle\right) & \text{if } \emptyset \in Y. \end{cases}$$

In particular, $\langle X \setminus \{\emptyset\} \rangle = |X|$ and $\langle X \rangle = |\operatorname{cone}(X)|$ where $\operatorname{cone}(X)$ is the simplicial complex $\Delta_0 * X$.

Any subset Y of X is a poset with respect to inclusion and we may construct the associated (normalized) order complex Y': The vertices of Y' are the elements of Y and the k-faces of Y' are flags $f_0 \subset f_1 \subset \cdots \subset f_k$ of Y-elements. If Y is a complex, Y' is the barycentric subdivision of $\operatorname{cone}(Y)$.

A complex and its barycentric subdivision have the same geometric realization, so if Y is a subcomplex of X, we have $|Y'| = \langle Y \rangle$. This identity is obtained by sending a vertex f of Y' to the barycenter of f in $\langle f \rangle$ if $f \neq \emptyset$ and the vertex corresponding to $\emptyset \in Y'$ to the vertex of the cone. For a general subset $Y \subseteq X$, we only know that $|Y'| \subseteq \langle Y \rangle$ inside $|X'| = \langle X \rangle$.

Lemma 7. *If $Y \subseteq X$, then $|Y'|$ is a deformation retract of $\langle Y \rangle$. In particular, both sets have the same cohomology.*

Proof. If $f \in \operatorname{cone}(X)$, then we may identify $\langle f \rangle \subset \langle X \rangle$ with the union of all $\langle F \rangle$ in $|X'|$ where $F = (f_0 \subset f_1 \subset \cdots \subset f_k)$ and $f_k = f$. Thus, $\langle Y \rangle$ as subset of $|X'|$ is the union of such $\langle F \rangle$ with $f_k \in Y$. For such an F let $F_Y \leq F$ be the maximal subflag consisting only of faces in Y. Now we can continuously retract $\langle F \rangle \cup \langle F_Y \rangle$ onto $\langle F_Y \rangle$ and this can be done simultaneously for all F belonging to the above union. $\square$

For a subset $Y \subseteq X$, let $C^\bullet(Y)$ be the cochain complex of Y'. We have an obvious inclusion of the k-flags in Y into $Y^{(k)}$ given by $f_0 \subset f_1 \subset \cdots \subset f_k \mapsto (f_0, f_1, \ldots, f_k)$. This induces a surjection of complexes $K^\bullet(Y) \to C^\bullet(Y)$ when Y has property U.

Lemma 8. *The surjection $K^\bullet(Y) \to C^\bullet(Y)$ is a quasi-isomorphism, i.e., it induces an isomorphism in cohomology.*

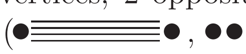
Proof. We will prove the dual statement in homology using the method of acyclic models (see e.g. [Spa66, 4.2]). If Y has property U, consider the simplicial complex Y^* where the vertices are the elements of Y and a set of vertices $\{f_0, \ldots, f_k\}$ is a face if $f_0 \cup \cdots \cup f_k \in Y$. If $C_\bullet(Y^*)$ is the chain complex of Y^*, then our $K^\bullet(Y)$ is the dual of $C_\bullet(Y^*) \otimes \mathbb{C}$, and we are finished if we can prove that $C_\bullet(Y^*)$ is chain equivalent to $C_\bullet(Y')$.
To this end, consider the set $\mathcal{Y}$ of U subsets of X as a category with inclusions as morphisms. Let the models in $\mathcal{Y}$ be $\mathcal{M} = \{\bar{f} \cap Y \mid f \in Y,\, Y \in \mathcal{Y}\}$. Finally, define the two functors from $\mathcal{Y}$ to chain complexes by $F'(Y) = C_\bullet(Y')$ and $F^*(Y) = C_\bullet(Y^*)$. We must show that both F' and F^* are free and acyclic with respect to these models. Now a basis element of $F'(Y)$, $(f_0 \subset f_1 \subset \cdots \subset f_k)$, comes from $F'(\overline{f_k} \cap Y)$ and a basis element $\{f_0, \ldots, f_k\}$ of $F^*(Y)$ comes from $F^*(\overline{f_0 \cup \cdots \cup f_k} \cap Y)$, so both functors are free. If $f \in Y$, one may check that $(\bar{f} \cap Y)^*$ is a simplex and $(\bar{f} \cap Y)'$ is a cone over the vertex $\{f\}$. Thus, in both cases the chain complexes are acyclic. $\square$

We may apply Lemma 7 and 8 to N_{a-b} and $\widetilde{N}_{a-b}$. Via the 5-Lemma, the result of Corollary 6 translates into

Theorem 9. *Assume $\mathbf{c} = \mathbf{a} - \mathbf{b}$ with $\mathbf{a}, \mathbf{b} \in \mathbb{N}^{n+1}$ having disjoint supports $a, b \subseteq [n]$. The homogeneous pieces in degree $\mathbf{c}$ of the cotangent cohomology of the Stanley-Reisner ring A_X vanish unless $\mathbf{b} \in \{0,1\}^{n+1}$, i.e., $\mathbf{b} = b$. Assuming $\mathbf{b} = b$, these modules only depend on the supports a, b, and we have isomorphisms*

$$T^i_{\mathbf{c}}(X) \;\simeq\; H^{i-1}(\langle N_{a-b}\rangle, \langle \widetilde{N}_{a-b}\rangle, \mathbb{C}) \;\; \textit{for } i = 1, 2$$

unless b consists of a single vertex. If b consists of only one vertex then the above formulae become true if we use the reduced cohomology instead.
Moreover, replacing $T^1_{\mathbf{c}}(X)$ by $\operatorname{Hom}(I_X, A_X)_{\mathbf{c}}$ creates true formulae without any need to distinguish between several cases.

Example 1.5 (continued) In the previous session of Example 1, we have seen that the pair $(N_{\emptyset-(yy')}, \widetilde{N}_{\emptyset-(yy')})$ equals (complete graph on 4 vertices, 2 opposite edges). Now, we may use that this is homotopy equivalent to (), i.e., $T^1_{\mathbf{c}}(D) = 0$, but $\dim_{\mathbb{C}} T^2_{\mathbf{c}}(D) = 4$.

The previous theorem also provides information about $T^0(X) := \operatorname{Der}_{\mathbb{C}}(A_X, A_X)$. While this module has already been described in [BS95], we would like to demonstrate its relation to our techniques.

Corollary 10. $T^0(X) = \bigoplus_{v=0}^n \mathfrak{a}_v\, \partial/\partial x_v$ *where $\mathfrak{a}_v$ is the ideal of A_X generated by the monomials x^a with $\overline{\operatorname{st}}(a, X) \subseteq \overline{\operatorname{st}}(v, X)$.*
In particular, $T^0(X)$ is generated, as a module, by $\delta_v := x_v\, \partial/\partial x_v$ if and only if every non-maximal $a \in X$ is properly contained in at least two different faces.

Proof. $T^0(X)$ is the kernel of $\mathrm{Der}_{\mathbb{C}}(\mathbb{C}[\mathbf{x}], A_X) \to \mathrm{Hom}(I_X, A_X)$. Hence, since $\widetilde{N}_{a-\{v\}} = \emptyset$, the previous theorem implies that an element $\mathbf{x}^{\mathbf{a}}\,\partial/\partial x_v$ belongs to $T^0(X)$ if and only if $H^0(\langle N_{a-\{v\}}\rangle, \mathbb{C}) = 0$, i.e., iff $N_{a-\{v\}} = \emptyset$. On the other hand, this means that, for every $f \in X$, the conditions $a \subseteq f$ and $v \notin f$ imply $f \cup v \in X$. Since the assumption $v \notin f$ can be omitted, this translates into $\mathrm{st}(a, X) \subseteq \overline{\mathrm{st}}(v, X)$. Finally, $T^0(X)$ is generated, as a module, by $\delta_v = x_v\,\partial/\partial x_v$ if and only if $\overline{\mathrm{st}}(a, X) \subseteq \overline{\mathrm{st}}(v, X)$ cannot happen for faces a with $v \notin a$. But this is equivalent to the condition formulated in the corollary. $\square$

4 Reduction to the $a = \emptyset$ case and localization

The set N_{a-b} is empty unless $b \neq \emptyset$ and $a \in X$ is a face. Moreover, by the next proposition, we may reduce all the calculations of N_{a-b} and $\widetilde{N}_{a-b}$, and therefore the T^i, to the case of $a = \emptyset$ on a smaller complex. See Example 17 for a demonstration of a consequent usage of this method.

Proposition 11. *$T^i_{\mathbf{c}}(X) = 0$ for $i = 1, 2$ unless $a \in X$ and $b \subseteq [\mathrm{lk}(a)]$. If $b \subseteq [\mathrm{lk}(a)]$, then the map $f \mapsto f \setminus a$ is a bijection $N_{a-b}(X) \xrightarrow{\sim} N_{\emptyset-b}(\mathrm{lk}(a))$ inducing isomorphisms $T^i_{\emptyset-b}(\mathrm{lk}(a)) \simeq T^i_{\mathbf{a}-b}(X)$ for $i = 1, 2$.*

Proof. First assume that there is a vertex $v \in b \setminus [\mathrm{lk}(a)]$. If $f \in N_{a-b}$, then $f \cup v \notin X$ (otherwise $a \cup v \in X$ and $v \in \mathrm{lk}(a)$). Thus, $N_{a-b} = \widetilde{N}_{a-b}$ unless $b = \{v\}$. If $b = \{v\}$, then $\widetilde{N}_{a-b} = \emptyset$ and $N_{a-b} = \mathrm{st}(a)$. Thus, $\langle N_{a-b}\rangle$ may be contracted to $\langle a\rangle$ and its reduced cohomology is trivial, so $T^i_{a-b}(X) = 0$ by Theorem 9.
It is a simple matter to check that the map between the N sets is a bijection (with inverse $g \mapsto g \cup a$) and that it restricts to a bijection of the $\widetilde{N}$ subsets. Since it clearly preserves inclusions, it induces a simplicial isomorphism on the complexes $N_{a-b}(X)' \simeq N_{\emptyset-b}(\mathrm{lk}(a))'$. From Lemma 7, it follows that we get an isomorphism in the relative cohomology. $\square$

Example 1.6 (continued) Assume that $a = \{x\}$ for D from Example 1. The link $\mathrm{lk}(x)$ equals the boundary of the rectangle $(yzy'z')$ plus the isolated point x'.
If $\#b \geq 3$, then there is always a proper subset $b' \subset b$ with $b' \notin X$. In particular, $N_{\emptyset-b}(\mathrm{lk}(x)) = \widetilde{N}_{\emptyset-b}(\mathrm{lk}(x))$, hence $T^1_{(x)-b}(D) = T^2_{(x)-b}(D) = 0$.
The case $\#b = 2$ does not provide any T^i either, but if $b = \{x'\}$, then $N_{\emptyset-(x')}(\mathrm{lk}(x))$ is the boundary of the rectangle and $\widetilde{N}_{\emptyset-(x')}(\mathrm{lk}(x)) = \emptyset$. Hence, $T^1_{(x)-(x')}(D) = 0$ and $\dim_{\mathbb{C}} T^2_{(x)-(x')}(D) = 1$. Similarily, we obtain $\dim_{\mathbb{C}} T^1_{(x)-*}(D) = 1$ and $T^2_{(x)-*}(D) = 0$ where $*$ stands for any of the vertices $y^{(')}$, $z^{(')}$.

The subsets $\langle N_{a-b}\rangle$ and $\langle \widetilde{N}_{a-b}\rangle$ are in general neither open nor closed in $|X|$. In the case $a = \emptyset$ though, we may find open sets retracting onto them. These sets are

easier to define, but they are not always easier to handle. However, their openess often allows use of standard tools for calculating cohomology. Let

$$U_b = U_b(X) := \{f \in X \mid f \cup b \notin X\}$$

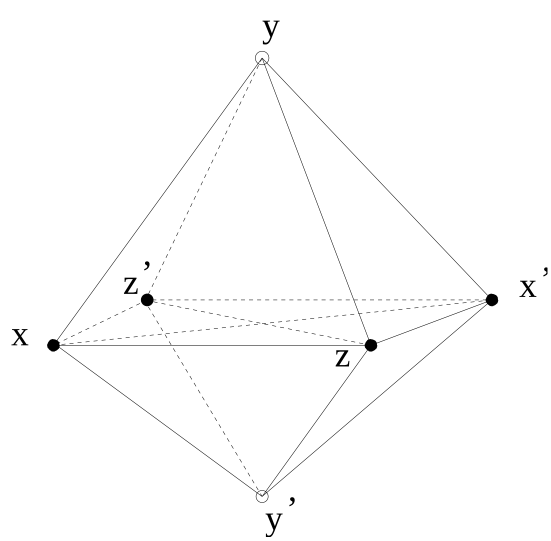
and

$$\widetilde{U}_b = \widetilde{U}_b(X) := \{f \in X \mid (f \cup b) \setminus \{v\} \notin X \text{ for some } v \in b\}\,.$$

Notice that $U_b = \widetilde{U}_b = X$ unless ∂b is a subcomplex of X. Moreover, if $\partial b \subseteq X$, then with $L_b := \bigcap_{b' \subset b} \mathrm{lk}(b', X)$ we have

$$X \setminus U_b = \begin{cases} \emptyset \\ \overline{\mathrm{st}}(b) \end{cases} \quad \text{and} \quad X \setminus \widetilde{U}_b = \begin{cases} \partial b * L_b & \text{if } b \text{ is a non-face,} \\ (\partial b * L_b) \cup \overline{\mathrm{st}}(b) & \text{if } b \text{ is a face.} \end{cases}$$

Example 1.7 (continued) Going back to the degree $\emptyset - (yy')$ from Example 1.5, we see that $D \setminus U_b$ is the (closed) subcomplex induced from the edge (yy'). On the other hand, L_b is the boundary of the rectangle $(xzx'z')$, hence $\partial b * L_b$ is the octahedron (without the diagonals). Thus, while U_b is D with the closed diagonal yy' being removed, $\widetilde{U}_b$ consists of the (disjoint) union of the open diagonals (xx') and (zz'). Comparing this with the easy $(N, \widetilde{N})$ in Example 1.5, we feel that one has to pay for the advantage of getting open sets.



$U_{(yy|)}$

Lemma 12. $H^\bullet(\langle N_{\emptyset - b}\rangle, \langle \widetilde{N}_{\emptyset - b}\rangle, \mathbb{C}) \simeq H^\bullet(\langle U_b\rangle, \langle \widetilde{U}_b\rangle, \mathbb{C})$.

Proof. If f is in U (respectively $\widetilde{U}$), then $f \setminus b$ is in N (respectively $\widetilde{N}$). Now, if $f \setminus b \neq \emptyset$ for all $f \in U$, then there is a standard retraction taking $\alpha \in \langle f \rangle$ to an $\alpha' \in \langle f \setminus b\rangle$ which fits together to make $(\langle N\rangle, \langle \widetilde{N}\rangle)$ a strong deformation retract of

$(\langle U\rangle, \langle\widetilde{U}\rangle)$. (See e.g. [Spa66, Proof of 3.3.11].)
If b is a face, then $f \setminus b = \emptyset$ is impossible, since then $f \cup b = b \in X$. If b is a non-face and $\emptyset \in \widetilde{N}$, i.e., $\partial b \not\subseteq X$, then all four spaces are cones and there is nothing to prove. If b is a non-face and $\emptyset \notin \widetilde{N}$, then both $\langle N\rangle$ and $\langle U\rangle$ are cones, so $H^i(\langle N\rangle, \langle\widetilde{N}\rangle) \simeq \widetilde{H}^{i-1}(\langle\widetilde{N}\rangle)$ and $H^i(\langle U\rangle, \langle\widetilde{U}\rangle) \simeq \widetilde{H}^{i-1}(\langle\widetilde{U}\rangle)$ and the above retraction works again. □

When we plug this into Theorem 9 and use Proposition 11 we get as a corollary the following description of the graded pieces of $T^i_{A_X}$.

Theorem 13. *The homogeneous pieces in degree $\mathbf{c} = \mathbf{a} - \mathbf{b}$ (with disjoint supports a and b) of the cotangent cohomology of the Stanley-Reisner ring A_X vanish unless $a \in X$ and $\mathbf{b} = b \neq \emptyset$. If these conditions are satisfied, we have isomorphisms*

$$T^i_{\mathbf{c}}(X) \;\simeq\; H^{i-1}\big(\langle U_b(\mathrm{lk}(a,X))\rangle,\, \langle\widetilde{U}_b(\mathrm{lk}(a,X))\rangle,\, \mathbb{C}\big) \;\; \textit{for } i = 1, 2$$

unless b consists of a single vertex. If b consists of only one vertex, then the above formulae become true if we use the reduced cohomology instead.

The reduction to the $a = \emptyset$ case also appears in a completely different context. One of the main issues of the paper [AC02] is the deformation theory of $\mathbb{P}(X) \subseteq \mathbb{P}^n$. The relation to the deformation theory of its affine charts $D_+(x_v)$ is governed by the localization maps $T^i_{A_X,0} \to T^i_{D_+(x_v)}$; here "0" is meant with respect to the usual $\mathbb{Z}$-grading of A_X, and the localization maps are obtained by dehomogenizing. Now, the point is that these affine charts also come from simplicial complexes. If $v \in [n]$, then $D_+(x_v) = \mathbb{A}(\mathrm{lk}(v,X))$, and we can use the techniques developed so far to describe the localization maps.

Remark. Although, strictly speaking, we should consider $\mathrm{lk}(v,X)$ as a subcomplex of Δ_n, the T^i depend only on the complex itself. In particular, when we look at graded parts, Proposition 11 shows that $T^i_{\mathbf{a}-b}(\mathrm{lk}(v)) = 0$ if a or b contain non-vertices of $\mathrm{lk}(v,X)$.

Lemma 14. *Let $\mathbf{c} = \mathbf{a} - b$ belong to degree 0, i.e., $\deg \mathbf{a} = \#b$ with $\deg$ denoting the sum of entries. Fix a vertex $v \in [n]$.*
(i) *Localization with respect to v maps $T^i_{\mathbf{a}-b} \subseteq T^i_{A_X,0}$ into the graded summand $T^i_{(\mathbf{a}\setminus v)-(b\setminus v)}(\mathrm{lk}(v,X)) \subseteq T^i_{D_+(x_v)}$ where $(\mathbf{a} \setminus v)$ means cancellation of the v entry.*
(ii) *The map of (i) is induced from $\psi_{a-b}(v) : N_{(a\setminus v)-(b\setminus v)}(\mathrm{lk}(v,X)) \to N_{a-b}(X)$ defined by $\psi_{a-b}(v)(g) = g \cup (v \setminus b)$. It is compatible with the $\widetilde{N}$ level.*
(iii) *If $v \in a$, then the localization map is an isomorphism in degree $\mathbf{a} - b$.*

Proof. It is straightforward to check that $\psi_{a-b}(v)$ is well defined and has the necessary properties to induce $\psi^*_{a-b}(v) : T^i_{\mathbf{a}-b}(X) \to T^i_{(\mathbf{a}\setminus v)-(b\setminus v)}(\mathrm{lk}(v,X))$ by Theorem 9. Moreover, it is clear that this means exactly dehomogenization with respect to v,

hence $\psi^*_{a-b}(v)$ coincides with the localization map.
Finally, to prove that $\psi_{a-b}(v) : g \mapsto g \cup v$ is an isomorphism in case of $v \in a$, we use that $\mathrm{lk}\big((a \setminus v), \mathrm{lk}(v, X)\big) = \mathrm{lk}(a, X)$ and apply Proposition 11 to both $N_{(a \setminus v)-b}(\mathrm{lk}(v, X))$ and $N_{a-b}(X)$. □

In fact, localizing with respect to *all* variables x_v with v running through the vertices of a given face $a \in X$ is induced by the map $\psi_{a-b}(a) : N_{\emptyset-b}(\mathrm{lk}(a, X)) \to N_{a-b}(X)$ sending g to $g \cup a$. This is the inverse of the a killing map of Proposition 11.

Theorem 15. *The maps* $T^i_{A_X,0} \to \bigoplus_{v\in[n]} T^i_{D_+(x_v)}$ *are injective for* $i = 1, 2$.

Proof. If a graded piece $T^i_{\mathbf{a}'-b'}(\mathrm{lk}(v))$ meets the image of $T^i_0(X) \to T^i(\mathrm{lk}(v))$, then its pre-image is a unique summand $T^i_{\mathbf{a}-b}(X) \subseteq T^i_0(X)$. Indeed, by Proposition 14, the only possibility to get degree 0 is $\mathbf{a} = \mathbf{a}' + (\#b' - \deg \mathbf{a}')\, v,\ b = b'$ if $\deg \mathbf{a}' \leq \#b'$ and $\mathbf{a} = \mathbf{a}',\ b = b' \cup v$ if $\deg \mathbf{a}' > \#b'$. This means that images of elements with different multidegree $\mathbf{c}$ cannot cancel each other, and it remains to consider the multigraded pieces

$$\bigoplus_{v\in[n]} \psi^*_{a-b}(v) :\ T^i_{\mathbf{a}-b}(X) \longrightarrow \bigoplus_{v\in[n]} T^i_{(\mathbf{a}\setminus v)-(b\setminus v)}(\mathrm{lk}(v))$$

with $\deg \mathbf{a} = \#b$. Since, by Proposition 14(iii), every summand $\psi^*_{a-b}(v)$ with $v \in a$ is an isomorphism, we obtain the injectivity of the above map whenever a has vertices at all. On the other hand, since $\deg \mathbf{a} = \#b$, the face a cannot be empty. □

5 Examples

First, in Examples 16 and 17, we present the complete treatment of the easiest X of all, the triangulations of 0- and 1-dimensional manifolds. While the dimensions of T^i for, say, the cone over the n-gon are already well known, we can demonstrate how the multigrading comes in. Moreover, for higher-dimensional examples like the surfaces in Example 18, the smaller ones are needed because they occur as links.

Example 16. Let $S^0 = \{\emptyset, 0, 1\}$ be the 0-dimensional complex consisting of two points only. It may be considered a triangulation of the 0-dimensional sphere.
If $b = \{0\}$, then $\widetilde{N}_{a-0} = \emptyset$ and $N_{a-0} = \{f \in S^0 \mid a \subseteq f,\ f \cup 0 \notin S^0\} = \{1\}$ for both possibilities $a = \emptyset$ or $a = \{1\}$. In particular, $T^1_{a-0}(S^0) = T^2_{a-0}(S^0) = 0$.
If $b = \{0, 1\}$, then $a = \emptyset$, hence $N_{\emptyset-\{0,1\}} = \{\emptyset\}$ and $\widetilde{N}_{\emptyset-\{0,1\}} = \emptyset$. This yields $T^2_{\emptyset-\{0,1\}}(S^0) = 0$, but $\boxed{T^1_{\emptyset-\{0,1\}}(S^0) \text{ is one-dimensional}}$.
How does this infinitesimal deformation perturb the S^0-equation $x_0 x_1 = 0$? If ε denotes the infinitesimal parameter from $\mathbb{C}[\varepsilon]/\varepsilon^2$, then one obtains $x_0 x_1 - \varepsilon = 0$.

Example 17. Denote by E_n the simplicial complex representing an n-gon with $n \geq 3$. Index the vertices cyclically with $0, \ldots, n-1$; all addition is done modulo n. First, we will show how to use the $a = \emptyset$ reduction from Proposition 11. If a is an edge, then $\operatorname{lk}(a, E_n) = \emptyset$, hence $T^i_{\mathbf{a}-b} = 0$. If $a = \{1\}$ is a vertex, then $\operatorname{lk}(a, E_n) = \{\emptyset, 0, 2\} \cong S^0$, hence from Example 16 we obtain $\boxed{\dim T^1_{\{1\}-\{0,2\}}(E_n) = 1}$ as the only non-trivial contribution; it translates into $x_0 x_2 - \varepsilon x_1^{\geq 1}$.
Let us now assume that $a = \emptyset$. If $\#b = 1$, i.e., b is a vertex, then $\widetilde{N}_{\emptyset-b} = \emptyset$, and $\langle N_{\emptyset-b}\rangle$ equals $|E_n|$ after removing the edges containing b. In particular, $\langle N_{\emptyset-b}\rangle$ is contractible, and the corresponding $T^i_{\emptyset-b}(E_n)$ are trivial. If b is an edge, then $\langle N_{\emptyset-b}\rangle$ looks similar, and $\langle \widetilde{N}_{\emptyset-b}\rangle$ equals $\langle N_{\emptyset-b}\rangle$ without the endpoints. We obtain $T^i_{\emptyset-b}(E_n) = 0$ for $n \geq 4$, but $\boxed{\dim T^1_{\emptyset-b}(E_3) = 1}$ yielding $x_0 x_1 x_2 - \varepsilon x_i$.
The final case is that $a = \emptyset$ and $b \notin E_n$. If $\#b \geq 3$, then, except for $n = 3$, the set b always contains proper subsets which are non-faces. In particular, $\widetilde{N}_{\emptyset-b} = N_{\emptyset-b}$, leading to $T^i_{\emptyset-b}(E_n) = 0$. The exception is $\boxed{\dim T^1_{\emptyset-\{0,1,2\}}(E_3) = 1}$ yielding $x_0 x_1 x_2 - \varepsilon$.
It remains to take two non-adjacent vertices for b, say $b = \{u, v\}$. Since $\emptyset \in N_{\emptyset-b}$, the set $\langle N_{\emptyset-b}\rangle$ is always a cone, hence contractible. In particular, $T^1_{\emptyset-b}(E_n) = 0$ if $\widetilde{N}_{\emptyset-b} \neq \emptyset$, and this is always the case except for $\boxed{\dim T^1_{\emptyset-\{v,v+2\}}(E_4) = 1}$; in terms of equations: $x_v x_{v+2} - \varepsilon$. On the other hand, the long exact cohomology sequence for the pair $(N_{\emptyset-b}, \widetilde{N}_{\emptyset-b})$ yields $T^2_{\emptyset-b}(E_n) = \widetilde{H}^0(\widetilde{N}_{\emptyset-b})$. Since the set $\langle \widetilde{N}_{\emptyset-b}\rangle$ equals $|E_n|$ with u and v and the adjacent edges being removed, we eventually obtain $\boxed{\dim T^2_{\emptyset-\{u,v\}}(E_n) = 1 \text{ whenever } |u - v| \geq 3}$.

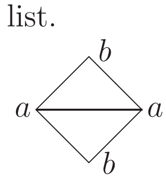
Adding up, we find that $T^2(E_n) = 0$ if $n \leq 5$, and that $\dim T^2(E_n) = n(n-5)/2$ if $n \geq 6$. In the latter case, we can even locate where the cup product takes place. Considering the coarse $\mathbb{Z}$-grading, we see that $T^1(E_n)$ spreads in degree ≥ -1, and $T^2(E_n)$ sits in degree -2. Hence, the cup product lives in the pieces $T^1_{-1} \times T^1_{-1} \to T^2_{-2}$ only. Using the $\mathbb{Z}^n$ multigrading, one obtains a finer result. The cup product splits into products

$$T^1_{\{v\}-\{v-1,v+1\}}(E_n) \ \times\ T^1_{\{v+1\}-\{v,v+2\}}(E_n) \ \longrightarrow\ T^2_{\emptyset-\{v-1,v+2\}}(E_n)$$

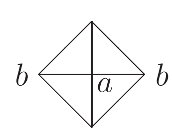
with all the three vector spaces being one-dimensional. See the continuation on p. 17 for more information.

Example 18. Let us look at the degree 0 deformations when X is the triangulation of a two-dimensional manifold. The non-zero multigraded pieces $T^1_{\mathbf{a}-b}(X)$ of $T^1_0(X)$ require $a \neq \emptyset$, hence they are induced from lower-dimensional links. They are all 1-dimensional and are gathered in the following list.

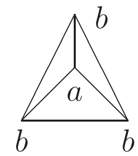
(i) a is an edge; $|a| = |b| = 2$.
$\mathbf{a} = a$.



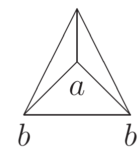
(ii) a is a vertex with valency 4;
$|a| = 1$, $|b| = 2$.
$\mathbf{a} = 2 \cdot a$.



(iii) a is a vertex with valency 3;
$|a| = 1$, $|b| = 3$;
$\mathbf{a} = 3 \cdot a$.



(iv) a is a vertex with valency 3;
$|a| = 1$, $|b| = 2$.
$\mathbf{a} = 2 \cdot a$.



The perturbation of the equations also comes from the corresponding one of the lower-dimensional link. E.g., denote the vertices in (i) such that $a = \{y, x_1\}$ and $b = \{x_0, x_2\}$. Then we obtain $x_0x_2 - \varepsilon x_1 y$.

Moreover, $T_0^2(X)$ is only present for a being a vertex of valency at least 6, $\mathbf{a} = 2 \cdot a$, and b consisting of two vertices having exactly a as a common neighbor.

Example 1.8 (finished) Finally, we conclude our running Example 1; the simplicial complex D is the octahedron plus the three diagonals (xx'), (yy'), and (zz').
First, if a is a non-empty face, then one has the following types of links: $\mathrm{lk}(xyz) = \emptyset$, $\mathrm{lk}(xx') = \emptyset$ (both yielding $T^1 = T^2 = 0$), $\mathrm{lk}(xy) = \{z, z', \emptyset\} \cong S^0$ yielding $\boxed{\dim T^1_{(xy)-(zz')}(D) = 1}$. Moreover, we studied the case $a = \{x\}$ in Example 1.6; the non-zero results were $\boxed{\dim T^2_{(x)-(x')}(D) = 1}$ and, for any $* \in \{y^{(\prime)}, z^{(\prime)}\}$, $\boxed{\dim T^1_{(x)-(*)}(D) = 1}$. In terms of equations, the T^1-contributions look like $yzz' - \varepsilon xy^2$ in the first case, and like $xx'y - \varepsilon x^2x'$, $xyy' - \varepsilon x^2y'$, $yy'z - \varepsilon xy'z$ in degree $(x) - (y)$.
Second, if $a = \emptyset$, then there are two major cases to distinguish. If b is not a face, then one knows in general that $\emptyset \in N_{\emptyset-b}(X)$, i.e., $\langle N_{\emptyset-b}(X)\rangle$ is contractible. Hence, by the long exact cohomology sequence for the pair $(N, \widetilde{N})$, $\dim T^1_{\mathbf{c}} = 1$ and $T^2_{\mathbf{c}} = 0$ if $\widetilde{N}_{\emptyset-b}(X) = \emptyset$, and, otherwise, $T^1_{\mathbf{c}} = 0$, $T^2_{\mathbf{c}} = \widetilde{H}^0(\widetilde{N}_{\emptyset-b}(X))$. In the case of $X = D$, this always yields $T^i_{\mathbf{c}} = 0$.
It remains to consider faces for b. While $b = \{x'\}$ or $\{x', y'\}$ do not yield any T^i, we obtained in Example 1.5 that $\boxed{\dim T^2_{\emptyset-(yy')}(D) = 4}$.

6 Appendix: The cup product

The cup product $T^1 \times T^1 \to T^2$ is an important tool to obtain more information about deformation theory than just the knowledge of the tangent or obstruction spaces T^i themselves. The associated quadratic form $T^1 \to T^2$ describes the equations of the versal base space up to second order.

In the case of Stanley-Reisner rings A_X, we only managed to get a nasty description of this product using the language of Proposition 4. We have not yet found a relation

to the geometry of the complex. However, since the cup product provides important information needed in some applications in [AC02], we have decided to present it in an appendix. It is suggested that overly-sensitive readers quit reading at this point.

The cup product can be defined in the following way (see [Lau79, 5.1.5]): Let $A = P/I$ with I generated by equations f^p. If $\varphi \in \operatorname{Hom}(I, A)$, lift the images of the f^p obtaining elements $\widetilde{\varphi}(f^p) \in P$. Given a relation $r \in R$, the linear combination $\langle r, \widetilde{\varphi}\rangle := \sum_p r_p\, \widetilde{\varphi}(f^p)$ vanishes in A, i.e. it is contained in I. If $\varphi, \psi \in \operatorname{Hom}(I, A)$ represent two elements of T^1, then we define for each relation $r \in R$

$$(\varphi \cup \psi)(r) := \psi(\langle r, \widetilde{\varphi}\rangle) + \varphi(\langle r, \widetilde{\psi}\rangle)\,. \tag{6.1}$$

This determines a well defined element of T^2. If $A = A_X$ is a Stanley-Reisner ring, then the cup product respects the multigrading, and, using Proposition 4, we can give a formula for $\cup : T^1_{\mathbf{a}^1-\mathbf{b}^1} \times T^1_{\mathbf{a}^2-\mathbf{b}^2} \to T^2_{\mathbf{a}-\mathbf{b}}$ with $\mathbf{a}-\mathbf{b} = \mathbf{a}^1 - \mathbf{b}^1 + \mathbf{a}^2 - \mathbf{b}^2$.

Proposition 19. *$\cup : T^1_{\mathbf{a}^1-\mathbf{b}^1} \times T^1_{\mathbf{a}^2-\mathbf{b}^2} \to T^2_{\mathbf{a}-\mathbf{b}}$ is determined by the following:*
(i) *If $b^1 \cap b^2 \neq \emptyset$, then $\cup = 0$.*
(ii) *If $b^1 \cap b^2 = \emptyset$, then $b = (b^1 \setminus a^2) \cup (b^2 \setminus a^1)$ and $a = (a^1 \setminus b^2) \cup (a^2 \setminus b^1) \cup (a^1_{\geq 2} \cup a^2_{\geq 2})$ with $a^\bullet_{\geq 2} := \{v \in [n] \mid a^\bullet_v \geq 2\}$ denoting the locus of higher multiplicities.*
(iii) *If $(f, g) \in N^{(2)}_{a-b}$, choose maximal subsets $d, e \subseteq b$ such that $f \cup (b \setminus d)$ and $g \cup (b \setminus e) \notin X$. If $\varphi \in T^1_{a^1-b^1}$ and $\psi \in T^1_{a^2-b^2}$, then the value of $(\varphi \cup \psi)(f, g)$ is*

$$\Big[\varphi\Big([f \setminus b^1] \cup [b^2 \setminus d]\Big) - \varphi\Big([g \setminus b^1] \cup [b^2 \setminus e]\Big)\Big] \cdot \psi\Big(f \cup g \cup a^2 \setminus b^2\Big)$$
$$+ \Big[\psi\Big([f \setminus b^2] \cup [b^1 \setminus d]\Big) - \psi\Big([g \setminus b^2] \cup [b^1 \setminus e]\Big)\Big] \cdot \varphi\Big(f \cup g \cup a^1 \setminus b^1\Big)$$

with φ, ψ defined to be zero on non-elements of X.

In fact, the maximality of d and e is not quite necessary. The point is to choose them non-empty whenever possible.

Proof. (i) We know that $\mathbf{b} \in \{0,1\}^n$ if $T^2_{\mathbf{a}-\mathbf{b}} \neq 0$. This would not be the case if $b^1 \cap b^2 \neq \emptyset$. Statement (ii) is a straightforward calculation.
For (iii), we must first recall what we did in Step 3 of the proof of Proposition 4. Antisymmetric functions $\mu(\bullet, \bullet)$ on the M level had been turned into antisymmetric functions $\lambda(\bullet, \bullet)$ on the N level via $\lambda(f, g) := \mu(m_f, m_g)$ with certain elements $m_\bullet \in \Phi^{-1}(\bullet)$. Hence, setting $p := m_f = f \cup (b \setminus d)$ and $q := m_g = g \cup (b \setminus e)$, we may compute the value of $(\varphi \cup \psi)(f, g)$ by applying the expression 6.1 on the relation $R_{p,q}$ described in the beginning of Section 2. Using $\widetilde{\varphi}(\mathbf{x}^p) = \varphi(\Phi_{a^1-b^1}(p)) \cdot \mathbf{x}^{p+\mathbf{a}^1-b^1}$, we obtain

$$\langle R_{p,q}, \widetilde{\varphi}\rangle = \Big[\varphi\big(\Phi_{a^1-b^1}(p)\big) - \varphi\big(\Phi_{a^1-b^1}(q)\big)\Big] \cdot \mathbf{x}^{(p\cup q)+\mathbf{a}^1-b^1}.$$

Plugging this into 6.1, we get

$$\begin{aligned}(\varphi \cup \psi)(f,g) &= \big[\varphi\big(\Phi_{a^1-b^1}(p)\big) - \varphi\big(\Phi_{a^1-b^1}(q)\big)\big] \cdot \psi\big(\Phi_{a^2-b^2}\big[\Phi_{a^1-b^1}(p\cup q)\big]\big) \\ &\quad + \big[\psi\big(\Phi_{a^2-b^2}(p)\big) - \psi\big(\Phi_{a^2-b^2}(q)\big)\big] \cdot \varphi\big(\Phi_{a^1-b^1}\big[\Phi_{a^2-b^2}(p\cup q)\big]\big)\,.\end{aligned}$$

To finish the proof, it is still necessary to take a closer look at the occurring arguments, i.e., to calculate

$$\begin{aligned}\Phi_{a^1-b^1}(p) &= \Phi_{a^1-b^1}\big(f \cup (b \setminus d)\big) \\ &= \big[f \cup (b^1 \setminus a^2 \setminus d) \cup (b^2 \setminus a^1 \setminus d) \cup a^1\big] \setminus b^1 \\ &= \big[f \cup (b^2 \setminus d)\big] \setminus b^1 \quad (\text{since } (a^1 \setminus b^2) \subseteq f \text{ and } a^1 \cap d = \emptyset) \\ &= (f \setminus b^1) \cup (b^2 \setminus d)\end{aligned}$$

and

$$\begin{aligned}\Phi_{a^2-b^2}\big[\Phi_{a^1-b^1}(p\cup q)\big] &= \Phi_{a^2-b^2}\big[\Phi_{a^1-b^1}\big((f\cup g) \cup (b \setminus (d\cap e))\big)\big] \\ &= \Phi_{a^2-b^2}\big[((f\cup g)\setminus b^1) \cup (b^2 \setminus (d\cap e))\big] \\ &= \big[((f \cup g \cup a^2) \setminus (b^1 \setminus a^2)\big] \setminus b^2 \\ &= \big[f \cup g \cup a^2\big] \setminus b^2 \quad (\text{since } (b^1 \setminus a^2) \cap (f\cup g) = \emptyset)\,.\end{aligned}$$

One can check that all the arguments of φ and ψ are in $N_{a^i-b^i}$ if they are in X. □

Example 17 (continued). We are going to calculate the cup product mentioned at the end of Example 17. With $f_1 := \{v-2\}$, $f_2 := \{v\}$ we choose representatives from both connected components of $\widetilde{N}_{\emptyset-\{v-1,v+2\}}$. Since $N_{\emptyset-\{v-1,v+2\}} = \widetilde{N}_{\emptyset-\{v-1,v+2\}} \cup \{\emptyset\}$, we obtain that $\lambda(f_1,\emptyset)$ and $\lambda(f_2,\emptyset)$ suffice to know about a function $\lambda : N_{\emptyset-\{v-1,v+2\}} \to \mathbb{C}$ from Proposition 4. Since T^2 results from dividing out a subspace, we obtain $[\lambda(f_1,\emptyset) - \lambda(f_2,\emptyset)]$ as the ultimate coordinate of $T^2_{\emptyset-\{v-1,v+2\}}(E_n)$. As auxillary elements $d,e \subseteq b$ (cf. Proposition 19), we may choose $d := \{v-1\}$ for both f_1 and f_2 and $e := \emptyset$ for the second argument $g := \emptyset$.
On the other hand, we know that $N_{\{v\}-\{v-1,v+1\}} = \{v\}$ and $N_{\{v+1\}-\{v,v+2\}} = \{v+1\}$ with $\widetilde{N} = \emptyset$ in both cases. Hence, the T^1 spaces are represented by maps φ and ψ yielding 1 on the faces $\{v\}$ and $\{v+1\}$, respectively. Applying Proposition 19, we find

$$\begin{aligned}(\varphi \cup \psi)(f_1,\emptyset) = {} & [\varphi(\{v-2,v,v+2\}) - \varphi(\{v,v+2\})] \cdot \psi(\{v-2,v+2\}) \\ & + [\psi(\{v-2,v+2\}) - \psi(\{v-1,v+1\})] \cdot \varphi(\{v-2,v\}) \;=\; 0\end{aligned}$$

and

$$\begin{aligned}(\varphi \cup \psi)(f_2,\emptyset) = {} & [\varphi(\{v,v+2\}) - \varphi(\{v,v+2\})] \cdot \psi(\{v+1\}) \\ & + [\psi(\{v+1\}) - \psi(\{v-1,v+1\})] \cdot \varphi(\{v\}) \;=\; 1\,.\end{aligned}$$

Thus, the cup product mentioned at the end of Example 17 yields $\varphi \cup \psi = 1$.

Corollary 20. *Let $n \geq 7$. If $t_1, \ldots, t_n$ denote the coordinates of $T^1_{-1}(E_n) = \oplus_{v \in \mathbb{Z}/n\mathbb{Z}} T^1_{\{v\}-\{v-1,v+1\}}(E_n)$, then the equations of the negative part of the base space S of the versal deformation of E_n are $t_v t_{v+1} = 0$ for $v \in \mathbb{Z}/n\mathbb{Z}$. In particular, E_n is not smoothable over S.*

Proof. Via the cup product, we see that each part $T^2_{\emptyset-\{v-1,v+2\}}(E_n)$ is responsible for $t_v t_{v+1}$ in the quadratic part of the obstruction equations. Moreover, since T^2 is concentrated in degree -2, no higher order obstructions involving *only* degree -1 deformations can appear, i.e., S is described by the desired equations.
Thus, in any flat deformation of degree -1, every other parameter must vanish. One directly checks that this means that any fiber is singular, in fact reducible. □

In contrast, if $n = 6$, then each of the three T^2-pieces is the common target of two different cup products. In particular, the negative part of the base space S is given by the equations $t_0t_1 - t_3t_4 = t_1t_2 - t_4t_5 = t_2t_3 - t_5t_0 = 0$. This yields the cone over the three-dimensional, smooth, projective toric variety induced by the prism over the standard triangle.

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